

BCS approximation to the effective vector vertex of superfluid fermions

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Abstract

We examine the effective interaction of nonrelativistic fermions with an external vector field in superfluid systems. In contrast to the complicated vertex equation, usually used in this case, we apply the approach which does not employ an explicit form of the pairing interaction. This allows to obtain a simple analytic expression for the vertex function only in terms of the order parameter and other macroscopic parameters of the system.

We use this effective vertex to analyze the linear response function of the superfluid medium at finite temperatures. At the time-like momentum transfer, the imaginary part of the response function is found to be proportional to V_F^4 , i.e. the energy losses through vector currents are strongly suppressed. As an application, we calculate the neutrino energy losses through neutral weak currents caused by the pair recombination in the superfluid neutron matter at temperatures lower than the critical one for the 1S_0 pairing. This approach confirms a strong suppression of the neutrino energy losses as predicted in Ref. [2].

I. INTRODUCTION

Interactions among fermions in a superdense medium substantially modify their coupling to external fields. This problem is of a great importance both for a laboratory superconductors and for a superfluid baryon matter of neutron stars, where the thermal breaking and recombination of Cooper pairs is considered as a possible dominant mechanism of the neutrino energy losses through neutral weak currents. The rate of reaction, in the 1S_0 superfluid neutron matter, was for the first time estimated in Ref. [1]. Recent analysis [2] has shown however that the approach used in this calculation contradicts the hypothesis of conservation of the vector current in weak interactions and therefore considerably overestimates the neutrino energy losses.

An alternative calculation of the same process was suggested in Ref. [3], where the Green function technique was used to connect the rate of neutrino emission with the current-current correlation function, also called the retarded weak polarization tensor, and the corrections due to the strong interactions have been incorporated in the weak vertex function. The estimate was based on the idea that the dominant response of nonrelativistic baryons is due to the coupling of the baryon density to neutrinos. In this case the relevant input for the calculation is the imaginary part of the temporal component of the retarded vector-vector polarization tensor, $\text{Im}\Pi^{00}(\omega, \mathbf{q})$, which was estimated by the authors in the kinematical domain $2\Delta < \omega < \infty$ and $\mathbf{q} = 0$, to obtain $\text{Im}\Pi^{00}(\omega > 2\Delta, \mathbf{0}) \neq 0$, where Δ is the superfluid energy gap, and $q = (\omega, \mathbf{q})$ is the transferred four momentum. It is easy to see, that this estimate contradicts the current continuity condition, $\omega\Pi^{00}(\omega, \mathbf{q}) = q_i\Pi^{i0}(\omega, \mathbf{q})$, which tells us $\Pi^{00}(\omega, \mathbf{0}) = 0$ when $\omega > 2\Delta$.

A one more calculation was recently done in Ref. [4] where the vertex renormalization is reduced to a study of the effective three-point vertices that sum-up particle-hole irreducible ladders in the scalar channel. The vector channel of the particle-hole interactions was parametrized by a constant (the lowest order Landau parameter) and the ideal summation is performed in the random phase approximation. The result of this calculation is also incorrect. Indeed, the whole longitudinal polarization function, as given by Eq. (35) and/or by Eq. (48) of this work, vanishes when the gap goes to zero. Thus, at temperatures higher the critical one for neutron pairing, both the real and imaginary part of the polarization function vanish instead of going over to those of normal (unpaired) Fermi liquid

A reasonable calculation of the medium response to external vector fields must satisfy the current continuity conditions. Neglect of this requirement leads to a considerable error in the estimate of the longitudinal response function. It is well known that the current conservation in superfluid fermionic systems is fulfilled in a nontrivial way. The interest to this problem has arisen many years ago in connection with the gauge invariance of the Bardeen-Cooper-Schrieffer (BCS) theory [5], [6], [7], [8], [9]. But the problem is topical up to now [2], [10], [11].

It was realized that the current continuity equation can be satisfied if the interaction among quasiparticles is incorporated in the coupling vertex to the same degree of approximation as the self-energy effect is included in the quasiparticle. This prescription yields the vertex equation explicitly incorporating two physical inputs – the pairing interaction and the energy gap which are not independent but connected by the gap equation. Because the realistic pairing interactions are momentum dependent [12], [13], [14], particularly due to a short-range repulsive core, the above set of equations admits only numerical solution.

On the other hand it is well known that the BCS theory of superfluidity needs the particular form of the pairing interaction only for a calculation of the order parameter (the energy gap Δ). The latter completely defines all the properties of a superconductor. Thus, it is reasonable to expect that, in the BCS approximation, the vertex function can be expressed only in terms of the order parameter, and other macroscopic parameters of the superfluid system, without of using of an explicit form of the pairing interaction. Up to now, the relevant solution of the vertex equation is obtained [7], [8] only for the case when both the transferred energy and momentum are smaller than the superfluid energy gap, $\omega, |\mathbf{q}| < \Delta$, and only for zero temperature, $T = 0$. In applications, however, we frequently deal with the time-like momentum transfer at finite temperature $T > 0$.

In this paper we focus on the analytic calculation of the corresponding effective vertex valid at finite temperatures and for ω and \mathbf{q} , satisfying only $\omega \ll \mu, |\mathbf{q}| \ll \mathbf{p}_F$, where $\mu = \mathbf{p}_F^2/2M^*$ and \mathbf{p}_F are the effective chemical potential and Fermi momentum of interacting fermions, respectively. Our calculation explicitly ensures the current conservation, using method that imposes this from the start. The essential idea of the method is as follows. We know that the external longitudinal field modifies the order parameter in the system. The corresponding self-consistent correction can be found directly from the condition of the current conservation. As was repeatedly discussed in the theory of superconductors, this

procedure is equivalent to the above summation of the vertex corrections [8], [10].

The vector vertex in the BCS approximation is to be considered as a starting point for incorporation of residual interactions in order to provide the current conservation in the superfluid system. In turn, the residual interactions, evaluated in the random phase approximation are known to take into account the effects of the medium polarization (see e.g. [14], [15]), which renormalizes the sound velocity and the normal part of the vector vertex. This is beyond the scope of our consideration.

The paper is organized as follows. Section 2 outlines some well known properties of normal nonrelativistic Fermi liquids at low temperatures and introduces the notations used in the following. In section 3, we derive the general expression for the effective vector vertex of electrically neutral fermions in the pair-correlated system and evaluate the obtained expression in the quasiparticle approximation. In section 4, we consider the linear response of the superfluid medium in the vector channel. As an application, in section 5, we evaluate the neutrino energy losses through neutral weak currents caused by the pair recombination of neutrons in the crust of neutron stars. Section 6 contains a short summary of our findings and the conclusion.

We use the system of units $\hbar = c = 1$, and the Boltzmann constant $k_B = 1$.

II. NORMAL FERMI LIQUID

In order to introduce the approach used in the following consideration this section outlines some well known properties of normal nonrelativistic Fermi liquids at low temperatures. The degenerate fermion systems are commonly treated in the quasiparticle approximation. Within this framework, near the Fermi surface, the inverse Green function of normal (unpaired) fermions takes the quasiparticle form

$$G_N^{-1}(\varepsilon, \mathbf{p}) \simeq \varepsilon - \xi_{\mathbf{p}}, \quad (1)$$

where

$$\xi_{\mathbf{p}} = \frac{\mathbf{p}^2}{2M^*} - \frac{\mathbf{p}_F^2}{2M^*} \simeq \frac{\mathbf{p}_F}{M^*}(\mathbf{p} - \mathbf{p}_F), \quad (2)$$

is the energy of a quasiparticle relative to the Fermi level. The effective mass M^* of a quasiparticle is connected to the Fermi momentum \mathbf{p}_F by means of the Fermi velocity $V_F = \mathbf{p}_F/M^*$ which is small in a nonrelativistic fermion system.

In the lowest-order expansion (1) the wave-function renormalization which accounts for the next-to-leading term in the expansion around the Fermi-energy is set to unity. For the same reason we neglect the imaginary part of the quasiparticle self-energy $\sim |\varepsilon| \varepsilon$.

It is convenient to work in the particle-hole picture (known as the Nambu-Gor'kov formalism) by introducing the quasiparticle fields as two-component (particle-hole) objects Ψ_p and using the Pauli matrices $\hat{\tau}_i$ ($i = 1, 2, 3$) operating in the Nambu-Gor'kov space (see e.g. [7]). Then the Hamiltonian of the system of free quasi-particles takes the form

$$H_0 = \sum_p \Psi_p^\dagger \xi_{\mathbf{p}} \hat{\tau}_3 \Psi_p \quad (3)$$

which corresponds to excited states in the particle-hole picture, while the ground state (vacuum) is the state where all negative energy "quasi-particles" ($\epsilon < 0$) are occupied and no positive energy particles exist.

In terms of these fields, the vector vertex of a quasiparticle becomes the diagonal matrix:

$$\hat{\gamma}_\mu(p+q, p) = \begin{pmatrix} \gamma_\mu(p+q, p) & 0 \\ 0 & -\gamma_\mu^\dagger(p+q, p) \end{pmatrix}, \quad (4)$$

where the quasiparticle component is given by

$$\gamma_\mu(p+q, p) = \left(1, \frac{1}{M^*} \left(\mathbf{p} + \frac{1}{2} \mathbf{q} \right) \right)$$

and the hole component is $\gamma_\mu^\dagger(p+q, p) = \gamma_\mu(-p, -p-q)$.

Using the Pauli matrices this can be recast as

$$\hat{\gamma}^\mu = \begin{cases} \hat{\tau}_3 & \text{if } \mu = 0, \\ \frac{1}{M^*} (\mathbf{p} + \frac{1}{2} \mathbf{q}) & \text{if } \mu = i = 1, 2, 3 \end{cases}. \quad (5)$$

The inverse quasiparticle propagator takes the following matrix form

$$\hat{G}_0^{-1}(\varepsilon, \mathbf{p}) = \varepsilon - \xi_{\mathbf{p}} \hat{\tau}_3. \quad (6)$$

III. SUPERFLUIDITY EFFECTS

In the particle-hole picture, the pairing represents a quasiparticle transition into a hole (and a correlated pair). Therefore the self-energy arising due to the pairing interaction has the off-diagonal form in the Nambu-Gor'kov space. In the absence of external fields this

anomalous self-energy is real-valued and can be written in terms of the energy gap Δ arising in the quasiparticle spectrum. Then the BCS inverse propagator of a quasiparticle has the following form (see e.g. [9]):

$$\hat{G}^{-1}(\varepsilon, \mathbf{p}) = \varepsilon - \xi_{\mathbf{p}} \hat{\tau}_3 - \Delta \hat{\tau}_1, \quad (7)$$

The effective vertex in the pair-correlated system can be written as

$$\hat{\Gamma}_{\mu} = \hat{\gamma}_{\mu} + \frac{\partial \hat{\Sigma}^{(1)}}{\partial V^{\mu}}. \quad (8)$$

where $\hat{\gamma}_{\mu}$ is the "normal" vertex, as given by Eq. (4), and $\hat{\Sigma}^{(1)}$ is the linear correction to the anomalous self-energy of a quasiparticle in the external vector field V^{μ} . Apparently this correction is due to varying of the energy gap in the external field. By taking the new gap in the form

$$\tilde{\Delta} = \Delta + \Delta^{(1)}(V^{\mu})$$

we can write

$$\hat{\Sigma}^{(1)} = \begin{pmatrix} 0 & \Delta^{(1)}(V^{\mu}) \\ \Delta^{*(1)}(V^{\mu}) & 0 \end{pmatrix}.$$

Now we use the fact that, in the case $\mathbf{q} \ll \mathbf{p}_F$, the linear effect of the weak field perturbation is to change the phase $\Phi(V^{\mu})$ but not the magnitude of the gap parameter [8], [10]. The field-induced correction to the amplitude of the gap is proportional to $\Delta/\mu \ll 1$ and thus can be neglected. We therefore write the energy gap as $\tilde{\Delta} = \Delta \exp i\Phi(V^{\mu})$, where Δ is the (real) gap in the absence of external field. Apparently the linear correction to the gap $\Delta^{(1)} = \tilde{\Delta} - \Delta$ is proportional to the external field and can be written as

$$\begin{aligned} \Delta^{(1)}(V^{\mu}) &\equiv \Delta \cdot (e^{i\Phi(V^{\mu})} - 1) \simeq i\Delta \cdot \Phi(V^{\mu}) \simeq -Q_{\mu}(q) V^{\mu}(q), \\ \Delta^{*(1)}(V^{\mu}) &\equiv \Delta \cdot (e^{-i\Phi(V^{\mu})} - 1) \simeq -i\Delta \cdot \Phi(V^{\mu}) \simeq Q_{\mu}(q) V^{\mu}(q), \end{aligned} \quad (9)$$

where $Q_{\mu}(q)$ is the unknown kernel connecting the correction to the energy gap with the external weak field. Then the effective vertex becomes

$$\hat{\Gamma}_{\mu}(p+q;p) = \begin{pmatrix} \gamma_{\mu}(p+q,p) & -Q_{\mu}(q) \\ Q_{\mu}(q) & -\gamma_{\mu}^{\dagger}(p+q,p) \end{pmatrix}, \quad (10)$$

and the problem reduces to calculation of the unknown vector function $Q_{\mu}(\omega, \mathbf{q})$.

The relation between the modified vertex Γ^μ and the quasiparticle propagator \hat{G} is given by the matrix Ward identity [16]:

$$q^\mu \hat{\Gamma}_\mu(p+q, p) = \hat{G}^{-1}(p+q) \hat{\tau}_3 - \hat{\tau}_3 \hat{G}^{-1}(p). \quad (11)$$

Applying this identity to expressions (10) and (7) we obtain the following well known relation [9]:

$$q_\nu Q^\nu(q) = -2\Delta. \quad (12)$$

For further progress, let us consider the medium response $\Pi^{\mu\nu}(q)$ to the external vector field. This calculation needs the quasiparticle propagator which, in contrast to the inverse propagator (7), must be uniquely defined at finite temperature. Since the matter is assumed in thermal equilibrium at some temperature, we employ the Matsubara calculation technique. In this case the quasiparticle propagator can be obtained by inverting Eq. (7), and substituting $\varepsilon = i\pi(2n+1)T$ with $n = 0, \pm 1, \pm 2, \dots$. Then the retarded polarization tensor $\Pi^{\mu\nu}(q)$ is given by the analytical continuation of the following Matsubara sums (see e.g. [16])

$$\begin{aligned} \Pi^{\mu\nu}(\omega_m, \mathbf{q}) = T \sum_{p_n} \int \frac{d^3p}{(2\pi)^3} \text{Tr} \left[\hat{\gamma}^\mu(p_n + \omega_m, \mathbf{p} + \mathbf{q}; p_n, \mathbf{p}) \hat{G}(p_n + \omega_m, \mathbf{p} + \mathbf{q}) \right. \\ \left. \times \hat{\Gamma}^\nu(p_n + \omega_m, \mathbf{p} + \mathbf{q}; p_n, \mathbf{p}) \hat{G}(p_n, \mathbf{p}) \right] \end{aligned} \quad (13)$$

onto the upper-half plane of the complex variable ω . Here $p_n = \pi(2n+1)T$, and $\omega_m = 2\pi mT$, with $m, n = 0, \pm 1, \pm 2, \dots$, are the fermionic and bosonic Matsubara frequency, respectively; and the quasiparticle propagator has the following form

$$\hat{G}(p_n, \mathbf{p}) \equiv \begin{pmatrix} G(p_n, \mathbf{p}) & F^\dagger(p_n, \mathbf{p}) \\ F(p_n, \mathbf{p}) & -G^\dagger(p_n, \mathbf{p}) \end{pmatrix}. \quad (14)$$

The particle and hole components are given by the on-diagonal elements, with $G^\dagger(p_n, \mathbf{p}) = G(-p_n, -\mathbf{p})$ while the off-diagonal elements, $F^\dagger(p_n, \mathbf{p}) = F(-p_n, -\mathbf{p})$, represent the anomalous contribution caused by the pairing interaction. According to Eq. (7), in the BCS approximation the components are given by the following expressions [17]:

$$\begin{aligned} G(p_n, \mathbf{p}) &= \frac{-ip_n - \xi_{\mathbf{p}}}{p_n^2 + \varepsilon_{\mathbf{p}}^2}, & F(p_n, \mathbf{p}) &= \frac{\Delta}{p_n^2 + \varepsilon_{\mathbf{p}}^2}, \\ F^\dagger(p_n, \mathbf{p}) &= F(p_n, \mathbf{p}), & G^\dagger(p_n, \mathbf{p}) &= \frac{ip_n - \xi_{\mathbf{p}}}{p_n^2 + \varepsilon_{\mathbf{p}}^2}, \end{aligned} \quad (15)$$

where $\varepsilon_{\mathbf{p}}$ is the energy of a quasiparticle

$$\varepsilon_{\mathbf{p}} = \sqrt{\xi_{\mathbf{p}}^2 + \Delta^2}. \quad (16)$$

It is important to notice that the magnitude of the energy gap in the above expressions depends on the temperature, $\Delta = \Delta(T)$. We assume that this function is found from the corresponding gap equation.

Using expressions (14), (10) in Eq. (13) gives

$$\Pi_{\mu\nu}(q) = \Lambda_{\mu\nu}(q) + \Lambda_{\mu}(q) Q_{\nu}(q), \quad (17)$$

where the one-loop integrals $\Lambda_{\mu\nu}(\omega, \mathbf{q})$ and $\Lambda_{\mu}(\omega, \mathbf{q})$ are defined as the analytical continuation of the following Matsubara sums:

$$\begin{aligned} \Lambda_{\mu\nu}(\omega_m, \mathbf{q}) = T \sum_{p_n} \int \frac{d^3 p'}{(2\pi)^3} & [\gamma_{\mu} \gamma_{\nu} G(p_n + \omega_m, \mathbf{p}' + \mathbf{q}) G(p_n, \mathbf{p}') \\ & + \gamma_{\mu}^{\dagger} \gamma_{\nu}^{\dagger} G^{\dagger}(p_n + \omega_m, \mathbf{p}' + \mathbf{q}) G^{\dagger}(p_n, \mathbf{p}') \\ & - (\gamma_{\mu} \gamma_{\nu}^{\dagger} + \gamma_{\mu}^{\dagger} \gamma_{\nu}) F(p_n, \mathbf{p}') F(p_n + \omega_m, \mathbf{p}' + \mathbf{q})] \end{aligned} \quad (18)$$

$$\begin{aligned} \Lambda_{\mu}(\omega_m, \mathbf{q}) = T \sum_{p_n} \int \frac{d^3 p'}{(2\pi)^3} & [\gamma_{\mu} G(p_n + \omega_m, \mathbf{p}' + \mathbf{q}) F(p_n, \mathbf{p}') \\ & + \gamma_{\mu}^{\dagger} F(p_n + \omega_m, \mathbf{p}' + \mathbf{q}) G^{\dagger}(p_n, \mathbf{p}') \\ & - \gamma_{\mu}^{\dagger} G^{\dagger}(p_n + \omega_m, \mathbf{p}' + \mathbf{q}) F(p_n, \mathbf{p}') \\ & - \gamma_{\mu} F(p_n + \omega_m, \mathbf{p}' + \mathbf{q}) G(p_n, \mathbf{p}')] . \end{aligned} \quad (19)$$

The unknown vector function $Q_{\nu}(q)$ can be found from the requirement that the polarization tensor (17) obeys the current conservation, $\Pi^{\mu\nu} q_{\nu} = 0$, and $q_{\nu} \Pi^{\nu\mu} = 0$. The first of these relations is satisfied automatically if the effective vertex $\hat{\Gamma}_{\mu}$ obeys the Ward identity (11), the second one is to be satisfied by a proper choice of the function $Q^{\mu}(q)$.

With the aid of relation (12) the above conditions can be written as the two coupled equations

$$\Lambda^{\mu\nu}(q) q_{\nu} - 2\Delta \Lambda^{\mu}(q) = 0, \quad (20)$$

$$q_{\nu} \Lambda^{\nu\mu}(q) + q_{\nu} \Lambda^{\nu}(q) Q^{\mu}(q) = 0. \quad (21)$$

Using the symmetry property, $q_{\nu} \Lambda^{\nu\mu} = \Lambda^{\mu\nu} q_{\nu}$, we find

$$Q^{\mu}(q) = -\frac{2\Delta}{q_{\lambda} \Lambda^{\lambda}(q)} \Lambda^{\mu}(q). \quad (22)$$

Inserting this in Eq. (10) we obtain the effective vector vertex in the following form:

$$\hat{\Gamma}_\mu(p+q;p) = \hat{\gamma}_\mu(p+q;p) - \frac{2i\Delta\hat{\tau}_2}{q_\lambda\Lambda^\lambda(q)} \Lambda_\mu(q). \quad (23)$$

Thus the BCS approximation reduces to the calculation of the vector function $\Lambda^\mu(q)$. In Eqs. (18), (19), the summation over the Matsubara fermionic frequency can be performed exactly. The subsequent analytical continuation yields

$$\begin{aligned} \Lambda_{00}(\omega, \mathbf{q}) = & - \int \frac{d^3\mathbf{p}}{2(2\pi)^3} \left[\left(1 - \frac{\xi_{\mathbf{p}}\xi_{\mathbf{p}+\mathbf{q}} - \Delta^2}{\varepsilon_{\mathbf{p}}\varepsilon_{\mathbf{p}+\mathbf{q}}} \right) \Phi_+ \right. \\ & \left. - \left(1 + \frac{\xi_{\mathbf{p}}\xi_{\mathbf{p}+\mathbf{q}} - \Delta^2}{\varepsilon_{\mathbf{p}}\varepsilon_{\mathbf{p}+\mathbf{q}}} \right) \Phi_- \right], \end{aligned} \quad (24)$$

$$\Lambda_0(\omega, \mathbf{q}) = - \int \frac{d^3\mathbf{p}}{2(2\pi)^3} \frac{\omega\Delta}{\varepsilon_{\mathbf{p}}\varepsilon_{\mathbf{p}+\mathbf{q}}} (\Phi_+ + \Phi_-), \quad (25)$$

$$\Lambda_i(\omega, \mathbf{q}) = - \frac{1}{M^*} \int \frac{d^3\mathbf{p}}{2(2\pi)^3} \left(\mathbf{p}_i + \frac{1}{2}\mathbf{q}_i \right) \frac{\Delta(\xi_{\mathbf{p}+\mathbf{q}} - \xi_{\mathbf{p}})}{\varepsilon_{\mathbf{p}}\varepsilon_{\mathbf{p}+\mathbf{q}}} (\Phi_+ + \Phi_-). \quad (26)$$

$$q_i\Lambda_i(\omega, \mathbf{q}) = - \int \frac{d^3\mathbf{p}}{2(2\pi)^3} \frac{\Delta(\xi_{\mathbf{p}+\mathbf{q}} - \xi_{\mathbf{p}})^2}{\varepsilon_{\mathbf{p}}\varepsilon_{\mathbf{p}+\mathbf{q}}} (\Phi_+ + \Phi_-). \quad (27)$$

where the following notations are introduced:

$$\Phi_\pm = \frac{\varepsilon_{\mathbf{p}+\mathbf{q}} \pm \varepsilon_{\mathbf{p}}}{(\varepsilon_{\mathbf{p}+\mathbf{q}} \pm \varepsilon_{\mathbf{p}})^2 - (\omega + i0)^2} \left(\tanh \frac{\varepsilon_{\mathbf{p}}}{2T} \pm \tanh \frac{\varepsilon_{\mathbf{p}+\mathbf{q}}}{2T} \right). \quad (28)$$

Eq. (23) is the general result valid for arbitrary momentum transfers. By the direct substitution one can verify that this effective vertex satisfies the Ward identity (11), providing the current conservation in the system.

As we can see, in the BCS approximation, only the longitudinal components of the vertex are modified because the transverse components of Λ_i vanish at the angle integration. Making use of this fact we need to calculate only the temporal component of the vertex because the remaining components can be found from the current continuity condition.

General expression for the vertex function, as given by Eqs. (23) - (28), has a complicated form. Significant simplification is possible, however, due to the fact that the quasiparticles are nonrelativistic, i.e $V_F \ll 1$ and because we are interested in $\mathbf{q} \ll \mathbf{p}_F$. The latter condition implies $\mathbf{q}/M^* = V_F\mathbf{q}/\mathbf{p}_F \ll V_F$ and we may neglect the quasiparticle recoil. Indeed, in the cases typical for astrophysical applications one has $V_F \sim 0.1$ while $\mathbf{q}/M^* \sim T/M^* \sim 10^{-4} \div 10^{-3}$. Under these conditions the contributions, caused by the recoil are estimated as $\mathbf{q}^2/(M^*\omega) \sim (\mathbf{q}/M^*)(q/\omega) \sim V_F(\mathbf{q}/\mathbf{p}_F)(q/\omega)$.

We have

$$\xi_{\mathbf{p}+\mathbf{q}} - \xi_{\mathbf{p}} = \mathbf{q}V_F \cos \theta + \frac{\mathbf{q}^2}{2M^*} = \mathbf{q}V_F \left(\cos \theta + \frac{\mathbf{q}}{2\mathbf{p}_F} \right) \quad (29)$$

where θ is the angle between \mathbf{p} and \mathbf{q} momenta. Insertion in Eq. (27) gives

$$q_i \Lambda_i(\omega, \mathbf{q}) = -\mathbf{q}^2 V_F^2 \int \left(\cos \theta + \frac{\mathbf{q}}{2\mathbf{p}_F} \right)^2 \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{\Delta}{2\varepsilon_{\mathbf{p}} \varepsilon_{\mathbf{p}+\mathbf{q}}} (\Phi_+ + \Phi_-).$$

In this expression, contributions due to the odd powers of cosine vanish after angle integrations. To the lowest order in small parameters we may take the integrand at $\mathbf{q} = 0$ to find the following relation

$$q_i \Lambda_i(\omega, \mathbf{q}) \simeq \frac{1}{\omega} \mathbf{q}^2 \frac{V_F^2}{3} \Lambda_0(\omega, \mathbf{0})$$

valid up to accuracy V_F^3 . Apparently to the same accuracy we can write

$$q_i \Lambda_i(\omega, \mathbf{q}) \simeq \frac{1}{\omega} \mathbf{q}^2 c_s^2 \Lambda_0(\omega, \mathbf{q}), \quad (30)$$

where $c_s \equiv V_F/\sqrt{3}$ is the sound velocity in the Fermi gas.

Insertion in Eq. (23) immediately gives

$$\hat{\Gamma}_0(p+q;p) \simeq \hat{\tau}_3 - 2i\Delta \hat{\tau}_2 \frac{\omega}{\omega^2 - \mathbf{q}^2 c_s^2}, \quad (31)$$

The space component of the longitudinal vertex can be found from the Ward identity:

$$\mathbf{q} \hat{\Gamma} \simeq \xi_{\mathbf{p}+\mathbf{q}} - \xi_{\mathbf{p}} - 2i\Delta \hat{\tau}_2 \frac{q^2 c_s^2}{\omega^2 - \mathbf{q}^2 c_s^2}. \quad (32)$$

It is necessary to stress that the above expressions are obtained from the exact expression Eq. (23) to accuracy $V_F^2 \ll 1$ valid in the nonrelativistic system. To this accuracy the temperature dependence of the effective vector vertex is restricted to the gap function $\Delta(T)$.

It is interesting to notice that the approximate Eqs. (31) and (32) reproduce formally the vertex functions as obtained in Ref. [7] for the case of small transferred energy and momentum, $\omega, \mathbf{q} \ll \Delta$, and zero temperature. In contrast, our analysis shows that Eqs. (31), (32) are valid under considerably weaker conditions $\omega \ll \mu \equiv \mathbf{p}_F^2/2M^*$, $\mathbf{q} \ll \mathbf{p}_F$ and for a time-like momentum transfer, $|\mathbf{q}| < \omega$, as well. The energy gap $\Delta = \Delta(T)$, in Eqs. (31), and (32), depends on the temperature below the critical temperature T_c . At temperatures $T > T_c$ the gap vanishes. In this case, Eqs. (31), and (32) give the vector vertex (5), as obtained for normal Fermi liquids.

As is well known, the poles in the vertex function at $\omega^2 = \mathbf{q}^2 c_s^2$ indicate the existence of the collective acoustic-like excitations in the condensate propagating with a velocity c_s .

The second term in (31), and (32) is the result of the coupling of $\hat{\tau}_3$ to the collective mode. This can be understood in the following way. $\hat{\Gamma}_\mu$ contains matrix elements for creation or annihilation of a pair out of the vacuum. This process can go through the virtual collective intermediate state. This collective contribution plays the important role in the conservation of the vector current in superfluid systems.

IV. LINEAR RESPONSE IN THE VECTOR CHANNEL

Having determined the effective vertices, we turn to evaluation of the complete polarization tensor, which according to Eqs. (17), (22) is given by

$$\Pi_{\mu\nu}(q) = \Lambda_{\mu\nu}(q) - \frac{2\Delta}{q_\lambda \Lambda^\lambda(q)} \Lambda_\mu(q) \Lambda_\nu(q), \quad (33)$$

With the aid of relation (20) this expression can be transformed to the form which explicitly exhibits the continuity of the current:

$$\Pi_{\mu\nu}(q) = \Lambda_{\mu\nu}(q) - \frac{1}{q^\lambda q^\delta \Lambda_{\lambda\delta}(q)} q^\alpha q^\beta \Lambda_{\mu\alpha}(q) \Lambda_{\beta\nu}(q).$$

Indeed, this form obeys $\Pi_{\mu\nu} q^\nu = q^\mu \Pi_{\mu\nu} = 0$. Using this fact we decompose the polarization tensor (33) into the sum of longitudinal (with respect to \mathbf{q}) and transverse components

$$\Pi^{\mu\nu}(q) = \Pi_l(q) \left(1, \frac{\omega}{\mathbf{q}} \mathbf{n}\right)^\mu \left(1, \frac{\omega}{\mathbf{q}} \mathbf{n}\right)^\nu + \Pi_t(q) g^{\mu i} (\delta^{ij} - \mathbf{n}^i \mathbf{n}^j) g^{j\nu}. \quad (34)$$

Here $\mathbf{n} = \mathbf{q}/q$ is a unit vector; the longitudinal and transverse polarization functions are defined as

$$\Pi_l(q) = \Pi^{00}(q), \quad \Pi_t(q) = \frac{1}{2} (\delta^{ij} - \mathbf{n}^i \mathbf{n}^j) \Pi^{ij}(q). \quad (35)$$

As it follows from the definition (18), $\Lambda_{\mu\nu}(\omega, \mathbf{q})$ represents the polarization tensor in the so-called one-loop approximation. The transverse external field does not influence the anomalous self-energy of a quasiparticle [17], therefore the transverse polarization function is given by the one-loop expression (36). The calculation yields the well known expression

$$\begin{aligned} \Pi_t(\omega, \mathbf{q}) = & -\frac{1}{4M^{*2}} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \mathbf{p}^2 \sin^2 \theta \left[\left(1 - \frac{\xi_{\mathbf{p}} \xi_{\mathbf{p}+\mathbf{q}} + \Delta^2}{\varepsilon_{\mathbf{p}} \varepsilon_{\mathbf{p}+\mathbf{q}}} \right) \Phi_+ \right. \\ & \left. + \left(1 + \frac{\xi_{\mathbf{p}} \xi_{\mathbf{p}+\mathbf{q}} + \Delta^2}{\varepsilon_{\mathbf{p}} \varepsilon_{\mathbf{p}+\mathbf{q}}} \right) \Phi_- \right]. \end{aligned} \quad (36)$$

Compare this to the linear response of superconducting electrons to external transverse electromagnetic field [17].

In contrast, the longitudinal external field varies the order parameter in the system. This physically manifests itself as the collective acoustic excitation in the condensate. Therefore the longitudinal polarization function has the additional contribution, which takes into account the collective motion of the condensate providing a continuity of the current in the system. We obtain

$$\Pi_l(q) = \Lambda_{00}(q) - \frac{2\Delta}{q\lambda\Lambda^\lambda(q)} \Lambda_0(q) \Lambda_0(q) \quad (37)$$

According to Eq.(30), up to accuracy V_F^3 this can be simplified as

$$\Pi_l(\omega, \mathbf{q}) \simeq \Lambda_{00}(\omega, \mathbf{q}) - \frac{2\Delta\omega}{\omega^2 - \mathbf{q}^2 c_s^2} \Lambda_0(\omega, \mathbf{q}) \quad (38)$$

or, identically,

$$\begin{aligned} \Pi_l(q) = & - \int \frac{d^3\mathbf{p}}{2(2\pi)^3} \left[\left(1 - \frac{\xi_{\mathbf{p}}\xi_{\mathbf{p}+\mathbf{q}} - \Delta^2}{\varepsilon_{\mathbf{p}}\varepsilon_{\mathbf{p}+\mathbf{q}}} \right) \Phi_+ - \left(1 + \frac{\xi_{\mathbf{p}}\xi_{\mathbf{p}+\mathbf{q}} - \Delta^2}{\varepsilon_{\mathbf{p}}\varepsilon_{\mathbf{p}+\mathbf{q}}} \right) \Phi_- \right] \\ & + \frac{2\Delta^2\omega^2}{\omega^2 - \mathbf{q}^2 c_s^2} \int \frac{d^3\mathbf{p}}{2(2\pi)^3} \frac{1}{\varepsilon_{\mathbf{p}}\varepsilon_{\mathbf{p}+\mathbf{q}}} (\Phi_+ + \Phi_-). \end{aligned}$$

By the use of the following trick

$$\frac{\omega^2}{\omega^2 - \mathbf{q}^2 c_s^2} \equiv 1 + \frac{\mathbf{q}^2 c_s^2}{\omega^2 - \mathbf{q}^2 c_s^2}$$

we can write

$$\begin{aligned} \Pi_l(q) = & - \int \frac{d^3\mathbf{p}}{2(2\pi)^3} \left[\left(1 - \frac{\xi_{\mathbf{p}}\xi_{\mathbf{p}+\mathbf{q}} + \Delta^2}{\varepsilon_{\mathbf{p}}\varepsilon_{\mathbf{p}+\mathbf{q}}} \right) \Phi_+ - \left(1 + \frac{\xi_{\mathbf{p}}\xi_{\mathbf{p}+\mathbf{q}} + \Delta^2}{\varepsilon_{\mathbf{p}}\varepsilon_{\mathbf{p}+\mathbf{q}}} \right) \Phi_- \right] \\ & + \frac{\mathbf{q}^2 c_s^2}{\omega^2 - \mathbf{q}^2 c_s^2} \int \frac{d^3\mathbf{p}}{2(2\pi)^3} \frac{2\Delta^2}{\varepsilon_{\mathbf{p}}\varepsilon_{\mathbf{p}+\mathbf{q}}} (\Phi_+ + \Phi_-). \end{aligned}$$

and substitute $\xi_{\mathbf{p}+\mathbf{q}}$ as given by Eq. (29). Then with the aid of the series expansion of the integrand up to accuracy $(\mathbf{q}V_F/\Delta)^3$, and the averaging over angles after some simplifications we arrive at the following result:

$$\begin{aligned} \Pi_l(q) = & -\frac{1}{2T} \frac{\mathbf{p}_F^2}{2\pi^2} \int d\mathbf{p} \left(1 - \frac{1}{2} \frac{\omega}{\mathbf{q}u_{\mathbf{p}}} \ln \frac{\omega + \mathbf{q}u_{\mathbf{p}}}{\omega - \mathbf{q}u_{\mathbf{p}}} \right) \left(1 - \tanh^2 \frac{\varepsilon_{\mathbf{p}}}{2T} \right) \\ & + \frac{\mathbf{q}^2 c_s^2}{(\omega + i0)^2 - \mathbf{q}^2 c_s^2} \frac{\mathbf{p}_F^2}{2\pi^2} \int d\mathbf{p} \frac{\Delta^2}{\varepsilon_{\mathbf{p}}^3} \tanh \frac{\varepsilon_{\mathbf{p}}}{2T} \\ & - \frac{\mathbf{q}^2 c_s^2}{(\omega + i0)^2 - \mathbf{q}^2 c_s^2} \frac{1}{2T} \frac{\mathbf{p}_F^2}{2\pi^2} \int d\mathbf{p} \frac{\Delta^2}{\varepsilon_{\mathbf{p}}^2} \\ & \times \left(1 - \frac{1}{2} \frac{\omega}{\mathbf{q}u_{\mathbf{p}}} \ln \frac{\omega + \mathbf{q}u_{\mathbf{p}}}{\omega - \mathbf{q}u_{\mathbf{p}}} \right) \left(1 - \tanh^2 \frac{\varepsilon_{\mathbf{p}}}{2T} \right), \end{aligned} \quad (39)$$

where

$$u_p \equiv V_F \frac{\xi_p}{\varepsilon_p}.$$

is the velocity of a quasiparticle.

As given by this expression, contributions into the longitudinal polarization of a superfluid fermion system arise both due to the motion of quasiparticles (the first line) and the collective motion of the condensate (the second line). There is also the mixed term given in the last two lines of Eq. (39).

At $T > T_c$ the energy gap vanishes and Eq. (39) reduces to the well known expression for the longitudinal polarization of a normal Fermi gas:

$$\Pi_l(q; T > T_c) = -\frac{p_F M^*}{\pi^2} \left(1 - \frac{1}{2} \frac{\omega}{q V_F} \ln \frac{\omega + q V_F}{\omega - q V_F} \right).$$

The quasiparticle contribution and the mixed term vanish at zero temperature, when all the fermions are paired and no quasiparticle excitations exist. Therefore, at zero temperature, the longitudinal polarization arises only due to the collective motion of the condensate. We obtain

$$\Pi_l(q; T = 0) = -\frac{p_F M^*}{\pi^2} \frac{q^2 c_s^2}{(\omega + i0)^2 - q^2 c_s^2}. \quad (40)$$

A. Imaginary part

The imaginary part of the polarization function deserves of a special consideration because it contains a complete information on the energy losses and scattering of external particles in the medium. Here we have to distinguish the space-like, $\mathbf{q}^2 > \omega^2$, and time-like, $\mathbf{q}^2 < \omega^2$, kinematical domains. One can easily see that the polarization function (39) has no imaginary part in the time-like kinematical domain. This due to the fact that the polarization function, as given by Eq. (39), is calculated to accuracy V_F^2 . As we shall see, in the time-like kinematical domain, the imaginary part is proportional to V_F^4 .

1. Space-like momentum transfer

In the space-like kinematical domain, $\mathbf{q}^2 > \omega^2$, the longitudinal polarization function has the imaginary part due to the pole contributions of the collective mode at $\omega^2 = \mathbf{q}^2 c_s^2$ and due to the Landau damping at $\omega < q u_p$. The corresponding expressions can be readily found

from Eq. (39). At finite temperature this expression has a cumbersome form and represents no practical interest in the nonrelativistic case. Therefore we shall restrict our consideration of the space-like momentum transfer to the case of zero temperature. From Eq. (40) we readily find

$$\text{Im } \Pi_l(q; T = 0) = -\frac{p_F M^*}{\pi^2} \mathbf{q}^2 c_s^2 \delta(\omega^2 - \mathbf{q}^2 c_s^2) \text{sign } \omega$$

Since no quasiparticles exist at zero temperature, the imaginary part arises only due to excitations of the collective motion of the condensate. This form satisfies the f -sum rule

$$\int_0^\infty \omega \text{Im } \Pi_l(q; T = 0) d\omega = \frac{p_F^3}{6\pi M^*} \mathbf{q}^2$$

2. Time-like momentum transfer

We focus now on the imaginary part of the retarded polarization function at the time-like momentum transfer, $\mathbf{q}^2 < \omega^2$, important for applications. As mentioned above, this imaginary part needs the calculation of corrections of the order V_F^4 .

At $\omega > 2\Delta$ and $\mathbf{q} < \omega$ the imaginary part in Eqs. (24)-(27) and (36) arises from the pole of the function Φ_+ at $\omega = \varepsilon_{\mathbf{p}+\mathbf{q}} + \varepsilon_{\mathbf{p}}$. Calculation of the imaginary part of the transverse polarization function (36) is trivial. We get

$$\text{Im } \Pi_t(\omega, \mathbf{q}) = \frac{1}{15\pi} V_F^4 p_F M^* \frac{\mathbf{q}^2 \Delta^2 \Theta(\omega^2 - \mathbf{q}^2)}{\omega^3 \sqrt{\omega^2 - 4\Delta^2}} \tanh \frac{\omega}{4T} + O\left(V_F^4 \frac{\mathbf{q}^2}{\mathbf{p}_F^2}, V_F^5\right) \quad (41)$$

where $\Theta(x)$ is the Heaviside step function.

Calculation of the imaginary part of the longitudinal polarization function is more complicated. First of all, we evaluate the one-loop integrals up to accuracy V_F^4 . This gives:

$$\Lambda_{00}(\omega, \mathbf{q}) \simeq a_0(\omega, \mathbf{q}) + a_2(\omega, \mathbf{q}) V_F^2 + O\left(V_F^3 \frac{\mathbf{q}^2}{\mathbf{p}_F^2}, V_F^5\right), \quad (42)$$

$$\Lambda_0(\omega, \mathbf{q}) \simeq b_0(\omega, \mathbf{q}) + b_2(\omega, \mathbf{q}) V_F^2 + O\left(V_F^3 \frac{\mathbf{q}^2}{\mathbf{p}_F^2}, V_F^5\right), \quad (43)$$

$$q_i \Lambda_i(\omega, \mathbf{q}) \simeq c_2(\omega, \mathbf{q}) V_F^2 + O\left(V_F^3 \frac{\mathbf{q}^2}{\mathbf{p}_F^2}, V_F^5\right), \quad (44)$$

where

$$\begin{aligned}
\text{Im } a_0 &= -\frac{\mathbf{p}_F M^*}{\pi} \frac{\Delta^2}{\omega \sqrt{\omega^2 - 4\Delta^2}} \tanh \frac{\omega}{4T}, \\
\text{Im } a_2 &= -\frac{\mathbf{p}_F M^*}{\pi} \frac{\mathbf{q}^2}{3\omega^2} \frac{\Delta^2}{\omega \sqrt{\omega^2 - 4\Delta^2}} \tanh \frac{\omega}{4T} \\
&\quad \times \left(\frac{2\omega^2 - 2\Delta^2}{\omega^2 - 4\Delta^2} - \frac{\omega^2 - 4\Delta^2}{16T^2} \left(1 - \tanh^2 \frac{1}{4T} \omega \right) \right), \tag{45}
\end{aligned}$$

$$\begin{aligned}
\text{Re } b_0 &= -\frac{\mathbf{p}_F M^*}{\pi^2} \mathcal{P} \int_{\Delta}^{\infty} d\varepsilon \frac{\Delta \omega}{\sqrt{\varepsilon^2 - \Delta^2}} \frac{1}{4\varepsilon^2 - \omega^2} \tanh \frac{\omega}{4T}, \\
\text{Im } b_0 &= -\frac{\mathbf{p}_F M^*}{2\pi} \frac{\Delta}{\sqrt{\omega^2 - 4\Delta^2}} \tanh \frac{\omega}{4T}, \\
\text{Im } b_2 &= -\frac{\mathbf{p}_F M^*}{2\pi} \frac{\mathbf{q}^2}{3\omega^2} \frac{\Delta}{\sqrt{\omega^2 - 4\Delta^2}} \tanh \frac{\omega}{4T} \\
&\quad \times \left(\frac{2\Delta^2 + \omega^2}{\omega^2 - 4\Delta^2} - \frac{(\omega^2 - 4\Delta^2)}{16T^2} \left(1 - \tanh^2 \frac{\omega}{4T} \right) \right), \tag{46}
\end{aligned}$$

$$\begin{aligned}
\text{Re } c_2 &= -\frac{2}{3} \mathbf{q}^2 \frac{\mathbf{p}_F M^*}{\pi^2} \mathcal{P} \int_{\Delta}^{\infty} d\varepsilon \frac{\Delta}{\sqrt{\varepsilon^2 - \Delta^2}} \frac{1}{4\varepsilon^2 - \omega^2} \tanh \frac{\omega}{4T}, \\
\text{Im } c_2 &= -\frac{\mathbf{p}_F M^*}{\pi} \frac{\mathbf{q}^2}{3\omega} \frac{\Delta}{\sqrt{\omega^2 - 4\Delta^2}} \tanh \frac{\omega}{4T}. \tag{47}
\end{aligned}$$

In the above, symbol \mathcal{P} means the principal value of the integral.

As already mentioned, at $\mathbf{q}^2 < \omega^2$, the imaginary part of Π_l is small as compared to the contributions evaluated in previous sections to accuracy V_F^3 . Therefore, instead of Eq. (38) we start from the exact form of the longitudinal polarization function, as given by Eq. (37) in order to obtain the following series expansion over powers of V_F

$$\Pi_l \simeq \left(a_0 - 2\frac{\Delta}{\omega} b_0 \right) + \left(a_2 - 2\frac{\Delta}{\omega} \left(b_2 + \frac{1}{\omega} c_0 \right) \right) V_F^2 - 2\frac{\Delta}{\omega^3} \frac{c_2^2}{b_0} V_F^4 \dots$$

Thus we have

$$\begin{aligned}
\text{Im } \Pi_l &\simeq \left(\text{Im } a_0 - 2\frac{\Delta}{\omega} \text{Im } b_0 \right) \\
&\quad + V_F^2 \left(\text{Im } a_2 - 2\frac{\Delta}{\omega} \text{Im } b_2 - 2\frac{\Delta}{\omega^2} \text{Im } c_0 \right) - V_F^4 \frac{2\Delta}{\omega^3} \text{Im} \left(\frac{c_2}{b_0} c_2 \right) \dots
\end{aligned}$$

Upon substituting Eqs. (45), (46), (47) in the first and second terms of this expression we find that these leading terms mutually cancel

$$\text{Im } a_0 - 2\frac{\Delta}{\omega} \text{Im } b_0 = 0,$$

$$\text{Im } a_2 - 2 \frac{\Delta}{\omega} \text{Im } b_2 - 2 \frac{\Delta}{\omega^2} \text{Im } c_0 = 0,$$

in accordance with the results obtained in previous sections.

In the third term we find identically

$$\frac{c_2}{\omega b_0} \equiv \frac{2}{3} \frac{q^2}{\omega^2},$$

and thus

$$\text{Im } \Pi_l \simeq -4 \frac{\Delta}{\omega^2} \frac{V^4}{3} \frac{q^2}{\omega^2} \text{Im } c_2$$

We finally obtain the following lowest-order expression

$$\text{Im} \Pi_l(\omega, \mathbf{q}) \simeq \frac{4}{\pi} c_s^4 \mathbf{p}_F M^* \frac{q^4 \Delta^2 \Theta(\omega - 2\Delta)}{\omega^5 \sqrt{\omega^2 - 4\Delta^2}} \tanh \frac{\omega}{4T} \quad (48)$$

valid at $\mathbf{q} < \omega$.

We see that, in comparison with the one-loop approximation, as given by Eqs. (42) and (45), the self-consistent result (48) is suppressed by many orders of magnitude. The suppression factor is $c_s^4 \sim 10^{-4}$.

V. NEUTRINO EMISSION AT THE PAIR RECOMBINATION

As an application of the obtained results we consider the neutrino-pair emission through neutral weak currents occurring at the recombination of quasiparticles into the condensate. The process is kinematically allowed due to the existence of a superfluid energy gap Δ , which admits the quasiparticle transitions with time-like momentum transfer $q = (\omega, \mathbf{q})$, as required by the final neutrino pair.

We consider the total energy which is emitted into neutrino pairs per unit volume and time which is given by the following formula (see details e.g. in [15]):

$$Q = \frac{G_F^2}{8} \sum_{\nu} \int \frac{\omega}{\exp(\frac{\omega}{T}) - 1} 2 \text{Im} \Pi_{\text{weak}}^{\mu\nu}(q) \text{Tr}(l_{\mu} l_{\nu}^*) \frac{d^3 q_1}{2\omega_1 (2\pi)^3} \frac{d^3 q_2}{2\omega_2 (2\pi)^3}, \quad (49)$$

where G_F is the Fermi coupling constant, l_{μ} is the neutrino weak current, and $\Pi_{\text{weak}}^{\mu\nu}$ is the retarded weak polarization tensor of the medium. The integration goes over the phase volume of neutrinos and antineutrinos of total energy $\omega = \omega_1 + \omega_2$ and total momentum $\mathbf{q} = \mathbf{q}_1 + \mathbf{q}_2$. The symbol \sum_{ν} indicates that summation over the three neutrino types has to be performed.

We shall consider the 1S_0 pairing of neutrons, which takes place in the superfluid crust of neutron stars. In this case, the total spin of a bound pair is zero and the axial-vector weak interaction occurs only due to small relativistic effects which are usually omitted in the calculations [1]. Therefore we concentrate on the vector part of the weak interactions. In the vector channel, the weak polarization tensor is given by $\Pi_{\text{weak}}^{\mu\nu}(q) = C_V^2 \Pi^{\mu\nu}(q)$, where C_V is the vector weak coupling constant of a neutron, and $\Pi^{\mu\nu}(q)$ stands for the polarization tensor as given by Eq. (34).

By inserting $\int d^4q \delta^{(4)}(q - q_1 - q_2) = 1$ in this equation, and making use of the Lenard's integral

$$\int \frac{d^3k_1}{2\omega_1} \frac{d^3k_2}{2\omega_2} \delta^{(4)}(q - q_1 - q_2) \text{Tr}(l^\mu l^{\nu*}) = \frac{4\pi}{3} (q_\mu q_\nu - q^2 g_{\mu\nu}) \Theta(q^2) \Theta(\omega),$$

where $\Theta(x)$ is the Heaviside step function, and $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ is the signature tensor, we can write

$$Q = \frac{1}{48\pi^4} G_F^2 C_V^2 \mathcal{N}_\nu \int_0^\infty d\omega \int_0^\omega d\mathbf{q} \, q^2 \frac{\omega}{\exp\left(\frac{\omega}{T}\right) - 1} \text{Im}\Pi^{\mu\nu}(q) (q_\mu q_\nu - q^2 g_{\mu\nu}),$$

where $\mathcal{N}_\nu = 3$ is the number of neutrino flavors.

Due to the current conservation $q_\mu q_\nu \text{Im}\Pi^{\mu\nu}(q) = 0$, and we find

$$Q = \frac{C_V^2 G_F^2}{48\pi^4} \mathcal{N}_\nu \int_0^\infty d\omega \int_0^\omega d\mathbf{q} \, q^2 \frac{(\omega^2 - \mathbf{q}^2) \omega}{\exp\left(\frac{\omega}{T}\right) - 1} \left[\left(\frac{\omega^2}{\mathbf{q}^2} - 1 \right) \text{Im}\Pi_l(q) + 2\text{Im}\Pi_t(q) \right],$$

where the functions $\text{Im}\Pi_{l,t}(q)$ are given by Eqs. (41) and (48).

After simple calculations the neutrino energy losses are found to be:

$$Q = \frac{536 \mathcal{N}_\nu}{42 \cdot 525 \pi^5} V_F^4 G_F^2 C_V^2 \mathbf{p}_F M^* T^7 y^2 \int_0^\infty \frac{z^4}{(e^z + 1)^2} dx \quad (50)$$

where $y = \Delta/T$, $z = \sqrt{x^2 + y^2}$.

We see that the neutrino radiation via the vector weak currents in the nonrelativistic system is suppressed by several orders of magnitude with respect to that predicted in [1]. The suppression factor is $\sim V_F^4$. This result is consistent with the result obtained in Ref. [2], by the use of the Fermi Golden rule.

VI. SUMMARY AND CONCLUSION

We have developed a unified approach for calculating the longitudinal response function of superfluid fermion system at finite temperatures. The effective vertex in the vector channel

is analytically expressed in terms of the order parameter, and other macroscopic parameters of the superfluid system, without of using of an explicit form of the pairing interaction. The general expression for the vertex function is given by Eq. (23). When evaluated to accuracy $V_F^2 \ll 1$, this expression becomes very simple and takes the form, as given by Eqs. (31), (32). The latter formally reproduce the result obtained in Ref. [7] for the case of zero temperature and small transferred energy and momentum. In contrast, our result is valid at finite temperatures and for ω and \mathbf{q} , satisfying only $\omega \ll \mu, |\mathbf{q}| \ll p_F$. As we found to this accuracy $V_F^2 \ll 1$ the temperature dependence of the effective vector vertex is restricted to the gap function $\Delta(T)$. This justifies the approach used in Ref. [2] where this form of the effective vertex was used in the calculation of the neutrino energy losses.

Special attention was paid to preserving the Ward identity for the vertex function and the relations for the polarization tensor components, that are implied by the conservation of the current. The transverse and longitudinal polarization functions, as represented by Eqs. (36), (39), are calculated up to accuracy V_F^2 .

The imaginary part of the response functions, as given by (36), (39), arises due to the pole contribution of the acoustic-like collective mode and due to the Landau damping. Both contributions vanish at the time-like momentum transfer, $\omega > 2\Delta$ and $\mathbf{q} < \omega$. In this kinematical domain, the imaginary part of the polarization functions is due to breaking of the correlated pairs and recombination of the quasiparticles back into the condensate. The corresponding analytical expressions are found to be proportional to V_F^4 , as given by Eqs. (41), (48).

As an application of obtained results we have calculated the neutrino-pair emission through neutral weak currents occurring at the recombination of quasiparticles into the condensate. We found the rate of neutrino energy losses in the vector channel is suppressed as compared to the one-loop results by a factor V_F^4 . The magnitude of the suppression is in accordance with the one predicted in Ref. [2]. Thus the neutrino emission from the singlet-correlated neutron matter is mainly due to the axial-vector currents. As is well known [1], the corresponding neutrino energy losses are proportional to V_F^2 and thus also small. Apparently, the modifications to the neutrino emission rate through the pair recombination process, as found above, call for a detail reassessment of their role in the late-time cooling

of neutron stars.

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